North Sydney Boys' High School

MATHEMATICS (HSC COURSE)

ASSESSMENT TASK 2 (2007)

Name | Mark: /60

QUESTION 1

(a) Write down the primitive function of $\sqrt{x}$. **2**

(b) Evaluate:

i. $\displaystyle\int_0^1 \left(5x^4 - 3x^2 + 7\right)\, dx$ **2**

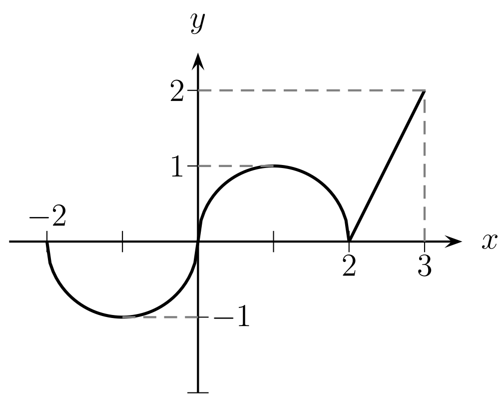
ii. $\displaystyle\int_{-1}^1 (2y-1)^5\, dy$ **3**

(c) The curve $y = f(x)$ has gradient function $\dfrac{dy}{dx} = 3 - 4x$. The curve passes through the point $(1, -1)$. Find the equation of the curve. **3**

(d) Find the value of A if **1**

$$2x^2 - 3x + 5 \equiv A(x-1)^2 + x + 3$$

(e) The diagram illustrates a function $y = f(x)$ for $-2 \le x \le 3$. It consists of 1 line segment & 2 semi-circles. **2**



Evaluate $\displaystyle\int_{-2}^{3} f(x)\, dx$.

QUESTION 2

(a) Sketch the locus of the point $P(x, y)$ which moves so that it is always a distance of 2 units from the point $(-2, 0)$. Hence write down its equation. **2**

(b) Solve the equation $x^4 - 7x^2 + 12 = 0$. **3**

(c) A point $Q(x, y)$ moves so that it is equidistant from the point $(1, 2)$ and the line $y = -2$. Describe the locus of point Q geometrically. **(Do not find its equation)**. **1**

(d) If α & β are roots of the equation $2x^2 - 7x - 5 = 0$, find the values of:

i. $\alpha + \beta$ **1**

ii. $\alpha\beta$ **1**

iii. $(\alpha + 1)(\beta + 1)$ **2**

iv. $(\alpha + 1)^{-1} + (\beta + 1)^{-1}$ **2**

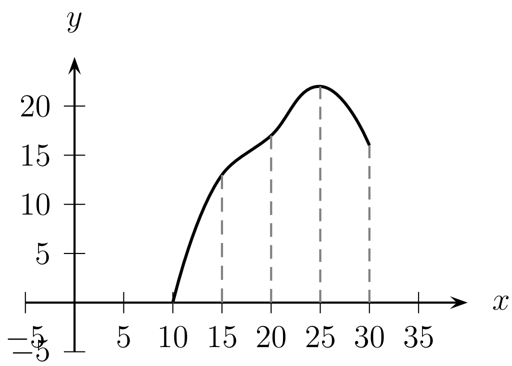
QUESTION 3

(a) Find:

i. $\int (2x - 1)(2x + 1)\,dx$ **2**

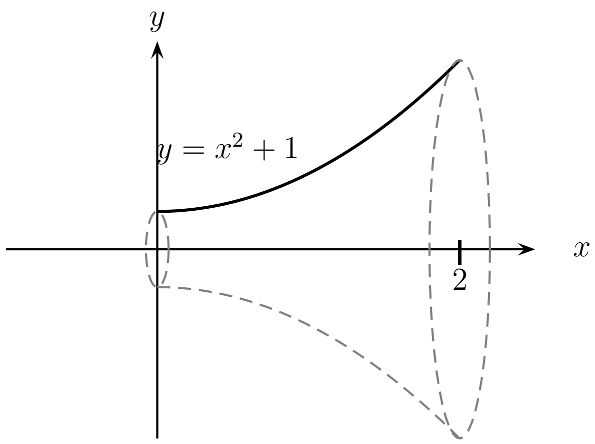
ii. $\int \left(\frac{2x^5 + 3}{x^5}\right) dx$ **2**

(b) Find the approximate area under the curve shown using the Trapezoidal Rule. **3**



x	10	15	20	25	30
y	0	13	17	22	16

(c) Find the volume of the solid of revolution shown below. **4**



QUESTION 4

(a) A parabola has vertex $V(3,1)$ and directrix $y = -1$. **2**
Find the equation of the parabola.

(b) Find the values of k in the quadratic equation $x^2 - 5x + k - 1 = 0$ if:

i. one root is equal to 2. **2**

ii. one root is the reciprocal of the other. **2**

(c) For the parabola $x^2 = 8y - 24$, find the coordinates of the focus. **2**

i. Find the equation of the locus of point P. **3**

ii. Describe this locus in geometrical terms, stating its important features. **2**

QUESTION 5

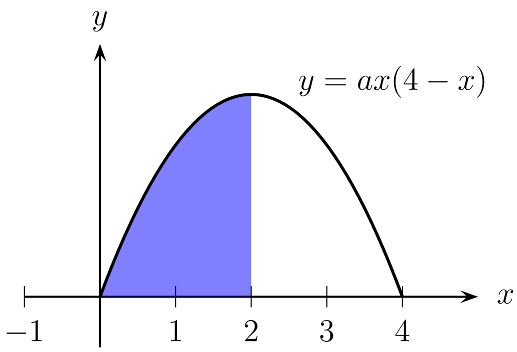
(a) Solve $(4+k)(1-k) < 0$. **1**

(b) For what values of k is the quadratic expression $kx^2 + 4x + (k+3)$ positive definite? (Hint: part (a) may be useful). **3**

(c) i. Differentiate $(x^2+3)^5$. **1**

ii. Hence, find $\int x(x^2+3)^4\,dx$. **1**

(d) The area of the shaded region is 40 square units. **3**



Find the value of a.

(e) If $\int_{-1}^{5} g(x)\,dx = 7$, find the value of:

i. $\int_{5}^{-1} g(x)\,dx$. **1**

ii. $\int_{-1}^{5} \left[3g(x) + 2\right] dx$. **2**

End of task

Brief solutions

QUESTION 1

(a) $\int x^{1/2}\,dx = \frac{2}{3}x^{3/2} + C$

(b) i. $\int_0^1 5x^4 - 3x^2 + 7\,dx$

$$= \left[x^5 - x^3 + 7x\right]_0^1$$
$$= 1 - 1 + 7 = 7$$

ii. $\int_{-1}^1 (2y-1)^5\,dy$

$$= \left[\frac{(2y-1)^6}{6\times 2}\right]_{-1}^1$$
$$= \frac{1}{12}\left((2-1)^6 - (2(-1)-1)^6\right)$$
$$= \frac{1}{12}(1-729) = -\frac{728}{12}$$
$$= -\frac{182}{3}$$

(c) $y' = 3 - 4x$.

$$y = \int 3 - 4x\,dx = 3x - 2x^2 + C$$

At $x = 1$, $y = -1$.

$$-1 = 3(1) - 2(1) + C$$
$$-1 = 1 + C$$
$$\therefore C = -2$$
$$\therefore y = 3x - 2x^2 - 2$$
$$= -2x^2 + 3x - 2$$

(d) $A = 2$:

$$2x^2 - 3x + 5 \equiv A(x-1)^2 + x + 3$$
$$= Ax^2 - 2Ax - A + x + 3$$
$$= Ax^2 - x(2A-1) + (3+A)$$

Equating coefficients, $A = 2$.

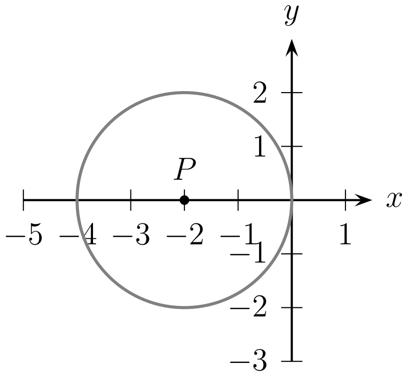
(e) Note that the areas of the semi-circles are offset and the remaining area is the area of the triangle with length $= 1$ and height $= 2$.

$$\int_{-2}^3 f(x)\,dx = \frac{1}{2}\times 2\times 1 = 1$$

QUESTION 2

(a) Using the distance formula to describe the distance from the point $P(x,y)$ to the point $(-2,0)$ being 2 units,

$$2 = \sqrt{(x+2)^2 + y^2}$$
$$4 = (x+2)^2 + y^2$$



which is a circle centred at $(-2,0)$ with radius 2.

(b) Letting $m = x^2$,

$$m^2 - 7m + 12 = 0$$
$$(m-4)(m-3) = 0$$
$$(x^2-4)(x^2-3) = 0$$
$$(x-2)(x+2)(x-\sqrt{3})(x+\sqrt{3}) = 0$$
$$\therefore x = \pm 2, \pm\sqrt{3}$$

(c) Parabola with focus at $(1,2)$, vertex $(1,0)$ and directrix $y = -2$.

(d) i. $\alpha + \beta = -\frac{b}{a} = \frac{7}{2}$

ii. $\alpha\beta = \frac{c}{a} = -\frac{5}{2}$

iii. $(\alpha+1)(\beta+1)$

$$= \alpha\beta + \alpha + \beta + 1$$
$$= \frac{7}{2} - \frac{5}{2} + 1$$
$$= 2$$

iv. $(\alpha+1)^{-1} + (\beta+1)^{-1}$

$$= \frac{1}{\alpha+1} + \frac{1}{\beta+1}$$
$$= \frac{\beta+1+\alpha+1}{(\alpha+1)(\beta+1)}$$
$$= \frac{\alpha+\beta+2}{(\alpha+1)(\beta+1)} = \frac{\frac{7}{2}+1}{2}$$
$$= \frac{11}{4}$$

QUESTION 3

(a) i. $\int (2x-1)(2x+1)\,dx$

$$= \int 4x^2 - 1\,dx$$
$$= \frac{4}{3}x^3 - x + C$$

ii. $\int \left(\frac{2x^5+3}{x^5}\right)dx$

$$= \int \frac{2\cancel{x^5}}{\cancel{x^5}} + \frac{3}{x^5}\,dx$$
$$= \int 2 + 3x^{-5}\,dx$$
$$= 2x - \frac{3}{4}x^{-4} + C$$

(b) $A \approx \frac{h}{2}\left(y_1 + \sum y_{\text{middle}} + y_\ell\right)$

$$= \frac{5}{2}\big(0 + 2(13+17+22) + 16\big)$$
$$= 300 \text{ units}^2$$

(c) Since $y = x^2 + 1$,

$$y^2 = (x^2+1)^2$$
$$= x^4 + 2x^2 + 1$$

$$V = \pi \int_a^b y^2\,dx$$
$$= \pi \int_0^2 x^4 + 2x^2 + 1\,dx$$
$$= \pi \left[\frac{1}{5}x^5 + \frac{2}{3}x^3 + x\right]_0^2$$
$$= \pi\left(\frac{32}{5} + \frac{16}{3} + 2\right)$$
$$= \frac{206\pi}{15}\text{ units}^3$$

QUESTION 4

(a) Focus-directrix form with $(3,1)$ being the vertex & $a = 2$ being the focal length as a is half of the distance between the focus & directrix.

$$(x-h)^2 = 4a(y-k)^2$$
$$\Rightarrow (x-3)^2 = 8(y-1)^2$$

(b) $x^2 - 5x + (k-1) = 0$.

i. $\alpha = 2$. Using the sum of roots,

$$\alpha + \beta = -\frac{b}{a}$$
$$2 + \beta = \frac{5}{1}$$
$$\therefore \beta = 3$$

Using the product of roots,

$$\alpha\beta = \frac{c}{a}$$
$$2 \times 3 = k - 1$$
$$\therefore k = 7$$

ii. $\alpha = \frac{1}{\beta}$. Using the product of roots,

$$\alpha\beta = \frac{c}{a}$$
$$1 = k - 1$$
$$\therefore k = 2$$

(c) $x^2 = 8y - 24$

$$\Rightarrow x^2 = 4 \times 2(y-3)$$

i.e. vertex is at $(0,3)$ and focal length $a = +2$. Hence the focus is at $(0,5)$.

(d) i. Applying the distance formula,

$$PA = \sqrt{(x+3)^2 + (y-1)^2}$$
$$PB = \sqrt{(x-3)^2 + (y+1)^2}$$

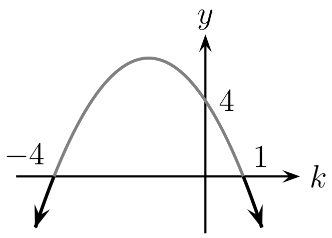
Applying the condition

$$PA^2 + PB^2 = 70$$
$$\overbrace{(x+3)^2 + (y-1)^2}^{PA^2}$$
$$+ \overbrace{(x-3)^2 + (y+1)^2}^{PB^2} = 70$$
$$(x^2 + \cancel{6x} + 9) + (y^2 - \cancel{2y} + 1)$$
$$+ (x^2 - \cancel{6x} + 9) + (y^2 + \cancel{2y} + 1) = 70$$
$$2(x^2+9) + 2(y^2+1) = 70$$
$$x^2 + y^2 + 10 = 35$$
$$\Rightarrow x^2 + y^2 = 25$$

ii. Circle of radius 5 centred at the origin.

QUESTION 5

(a) i. Sketching $y = (4+k)(1-k)$,



ii. In $y = kx^2 + 4x + (k+3)$,

$$\begin{aligned}\Delta &= b^2 - 4ac \\ &= 4^2 - 4k(k+3) \\ &= 4(4 - k^2 + 3k)\end{aligned}$$

$y = kx^2 + 4x + (k+3)$ is positive definite when $k > 0$ & $\Delta < 0$.

$$\begin{aligned}4(-k^2 + 3k + 4) &< 0 \\ (4+k)(1-k) &< 0\end{aligned}$$

Using Question 5(a)i,

$$k > 1 \quad k < -4$$

But it was established that $k > 0$. Hence $k > 1$ for the quadratic to be positive definite.

(b) i. Differentiating by the chain rule,

$$\begin{aligned}\frac{d}{dx}\left(x^2+3\right)^5 &= 5 \times 2x\left(x^2+3\right)^4 \\ &= 10x\left(x^2+3\right)^4 \end{aligned} \qquad (5.1)$$

ii. Using Equation (5.1),

$$\begin{aligned}\int x(x^2+3)^4\,dx &= \frac{1}{10}\int 10x\left(x^2+3\right)^4\,dx \\ &= \frac{1}{10}\left(x^2+3\right)^5 + C\end{aligned}$$

(c) Integrating,

$$\begin{aligned}40 &= \int_0^2 4ax - ax^2\,dx \\ &= \left[2ax^2 - \frac{a}{3}x^3\right]_0^2 \\ &= (2a\cdot 4) - \left(\frac{a}{3}\cdot 8\right) \\ &= \frac{16a}{3} \\ \therefore a &= \frac{120}{16} = \frac{15}{2}\end{aligned}$$

(d) i. $\displaystyle\int_5^{-1} g(x)\,dx = -\int_{-1}^5 g(x)\,dx = -7$.

ii. $\displaystyle\int_{-1}^5 \big(3g(x)+2\big)\,dx$

$$\begin{aligned}&= \int_{-1}^5 3g(x)\,dx + \int_{-1}^5 2\,dx \\ &= 3\int_{-1}^5 g(x)\,dx + \int_{-1}^5 2\,dx \\ &= 3\times 7 + 2\times 6 \\ &= 33\end{aligned}$$